MPP-2007-55
LMU-ASC 30/07

Thresholds for Intersecting D-branes Revisited

Nikolas Akerblom[1], Ralph Blumenhagen[1], Dieter Lüst[1,2],
Maximilian Schmidt-Sommerfeld[1]

[1] *Max-Planck-Institut für Physik, Föhringer Ring 6, 80805 München, Germany*

[2] *Arnold-Sommerfeld-Center for Theoretical Physics, Department für Physik, Ludwig-Maximilians-Universität München, Theresienstraße 37, 80333 München, Germany*

`akerblom, blumenha, luest, pumuckl@mppmu.mpg.de`

Abstract

Gauge threshold corrections for intersecting D6-brane string models on toroidal orbifold backgrounds are reconsidered. Both by dimensionally regularising the appearing open string one-loop diagrams in tree-channel as well as by zeta-function regularisation of the corresponding loop-channel one-loop diagrams, we arrive at a result which takes into account the infrared divergence from the contribution of the massless states in the running of the gauge coupling constant as well as the contribution of states, which become light in certain regions of the moduli space.

Contents

1 Introduction

Models based on orientifolds of type IIA/B string theory [1, 2, 3, 4] have become an alternative to heterotic constructions in studying low energy effects of string theory. In contrast to the latter, different gauge groups are usually localised on different brane stacks which implies that the tree level gauge couplings vary from stack to stack. For a high string scale, this can be at variance with gauge coupling unification as it appears in the MSSM. Therefore, corrections to the gauge coupling constants in intersecting D6-brane models are quite important if one wants to build semi-realistic models and eventually make contact with experiment.

Recently, it was found that the very same quantities also appear in the computation of D-instanton corrections to such intersecting D6-brane models. In this context they quantify the one-loop determinants of the fluctuations around the E2-instanton [5, 6, 7][1].

For intersecting D6-branes on toroidal backgrounds, these threshold corrections have been computed explicitly in [15]. (For a calculation in type I models see [16, 17, 18, 19].) These results were generalised to Gepner models in [20]. In this paper we would like to revisit the actual computation performed in [15]. Special care has to be taken of the two different divergences appearing in the relevant annulus and Möbius diagrams. Namely, there are infrared divergences stemming from massless open string modes as well as ultraviolet divergences, which are due to massless closed string tadpoles and which sum to zero upon invoking the tadpole cancellation condition. We use two different regularisation methods. First, we compute in tree channel, where the divergence due to the tadpole can be extracted explicitly. The infrared divergence is then taken care of by dimensional regularisation. Second, we perform the computation entirely in loop channel. Here, the infrared divergence is manifest and can be subtracted

[1]For related recent work on D-instanton effects see [8, 9, 10, 11, 12, 13, 14].

explicitly and the ultraviolet divergence is dealt with using zeta-function regularisation of divergent series. Both methods give the same result in sectors preserving $\mathcal{N} = 1$ supersymmetry[2]. However, this result differs slightly from the one given in [15]. The aim of this letter is to clarify this subtle issue.

In section 2 we display the method used to derive the one loop corrections to the gauge couplings. In section 3 we perform the actual calculations (in tree channel) and in section 4 we discuss the results and their relation to [15]. The loop channel calculation is sketched in the appendix.

2 One-loop thresholds for intersecting D6-branes on $\mathbb{T}^6$

The one-loop corrections to the gauge coupling constants can be computed by means of the background field method, which essentially amounts to computing the partition function in the presence of a magnetic field in the four-dimensional space-time.

The gauge coupling constants of the various gauge group factors G_a, up to one loop, have the following form

$$\frac{1}{g_a^2(\mu)} = \frac{1}{g_{a,\text{string}}^2} + \frac{b_a}{16\pi^2}\,\ln\left(\frac{M_s^2}{\mu^2}\right) + \Delta_a\,, \tag{2.1}$$

where b_a is the beta function coefficient. The first term corresponds to the gauge coupling constant at the string scale, which contains the tree-level gauge coupling as well as universal contributions at one-loop, the second term gives the usual one-loop running of the coupling constants, and the third term denotes the one-loop string threshold corrections originating from integrating out massive string excitations. All terms are encoded in the aforementioned partition functions and can therefore be determined by calculating all annulus and Möbius diagrams with at least one boundary on the brane where the gauge group factor G_a is localised. As we will discuss at the end of this letter, there is a subtle issue concerning the contribution of massive states in Δ_a, which become lighter than the string scale M_s for small intersection angles.

For the contribution of an annulus diagram to the threshold corrections for relatively supersymmetric intersecting branes, the background field method gives the general expression

$$T^A(\mathrm{D6}_a, \mathrm{D6}_b) = \int_0^\infty \frac{dt}{t} \sum_{\alpha,\beta\neq(\frac{1}{2},\frac{1}{2})} (-1)^{2(\alpha+\beta)}\, \frac{\vartheta''[{\alpha \atop \beta}](it)}{\eta^3(it)}\; A_{ab}^{\mathrm{CY}}[{\alpha \atop \beta}](it)\,, \tag{2.2}$$

[2]This is actually also true for $\mathcal{N} = 2$ sectors but, as the results agree with [15], the derivation will not be displayed here.

where A_{ab}^{CY} denotes the annulus partition function in the (ab) open string sector of the internal $\mathcal{N}=2$ superconformal field theory describing the Calabi-Yau manifold. So far, the gauge thresholds can only be explicitly computed for toroidal orbifold or Gepner models [20].

Let us from now on specialise to the case of the toroidal $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. In general, besides the O6-planes, an intersecting D-brane model contains various stacks of D6-branes wrapping factorisable supersymmetric three-cycles defined by three pairs of wrapping numbers (m_a^I, n_a^I), $I = 1, 2, 3$. On each D6-brane we assume a gauge symmetry $U(N_a)$ and we are interested in the one-loop gauge threshold corrections to the gauge couplings of these $U(N_a)$ gauge symmetries. These one-loop thresholds are given by annulus and Möbius diagrams with one boundary on the $U(N_a)$ brane.

3 Thresholds for $\mathcal{N}=1$ sectors

We now come to the actual regularisation of amplitudes. We take as our starting point the raw amplitudes found in [15]. For the annulus diagram in an $\mathcal{N}=1$ sector, the expression to be examined is

$$\begin{aligned} T^A(\mathrm{D6}_a, \mathrm{D6}_b) &= \frac{iI_{ab}N_b}{2\pi} \int_0^\infty \frac{dt}{t} \sum_{I=1}^3 \frac{\vartheta_1'}{\vartheta_1}\Big(\tfrac{i\theta_{ab}^I t}{2}, \tfrac{it}{2}\Big) \\ &= -\frac{I_{ab}N_b}{\pi} \int_0^\infty dl \sum_{I=1}^3 \frac{\vartheta_1'}{\vartheta_1}(-\theta_{ab}^I, 2il)\,, \end{aligned} \tag{3.1}$$

where I_{ab} is the intersection number, N_b is the number of branes on stack b and $l = 1/t$. Additionally, $\pi\theta_{ab}^I$ is the intersection angle of branes a and b on the I'th torus. Supersymmetry then imposes the 'angle condition'

$$\sum_{I=1}^3 \theta_{ab}^I = 0\,, \tag{3.2}$$

which we assume to be fulfilled.

As it stands, (3.1) is divergent. As mentioned in the introduction, there are two sources for this divergence: In the q-series of the l-channel integrand there is a constant (i.e. q^0-) term, giving us something proportional to $\int dl$. Even if this is subtracted (or thought of as taken care of by tadpole cancellation) there remains a divergence from $l \to 0$. The latter is the same as the divergence for $t \to \infty$ coming from the constant term in the loop channel. It is therefore seen to be a logarithmic divergence which arises from the massless open string states and encodes the one-loop running of the gauge couplings. Therefore, it will later on be replaced by $\ln\left(\frac{M_s^2}{\mu^2}\right)$.

Since, following [15], we impose the tadpole cancellation condition, we do not worry about the q^0-divergence, but clearly something needs to be done about the remaining logarithmic divergence. There are at least two ways to proceed. One is to subtract this divergence in the t-channel and another is to employ 'dimensional regularisation'. If one wants to extract the tadpole divergence manifestly, which is only possible in tree channel, at least the large l (i.e. small t) part of the amplitude has to be calculated in tree channel. This means that one can only work in the large t regime of the loop channel or, in other words, the t-integration has to be cut off at a finite lower limit. Since the computation is then difficult to do analytically, we shall carry out the tree channel computation using 'dimensional regularisation'[3].

By carrying out the instructions given just before (A.2)[4] it is easy to derive the following formula:

$$-\frac{1}{\pi}\int_0^\infty dl\,\frac{\vartheta_1'}{\vartheta_1}(\theta,2il) = -\cot(\pi\theta)\int_0^\infty dl + A\,, \tag{3.3}$$

where

$$A = 4i\int_0^\infty dl \sum_{m,n=1}^{\infty} \exp(-4\pi lmn)\,\sinh(2\pi i\theta m)\,. \tag{3.4}$$

When taking the sum over the θs, (3.3) gives us (3.1) (up to prefactors), so we might as well regularise (3.3). The first term in (3.3) is the tadpole and the second needs to be regularised. To this end, we let $\int dl \to \int dl\, l^\epsilon$, where ϵ is a (small) positive number. In loop channel this amounts to $\int \frac{dt}{t} \to \int \frac{dt}{t^{1+\epsilon}} \simeq \frac{1}{\epsilon}$. In analogy to the heterotic string [21], it is therefore justified to later substitute $\ln\left(\frac{M_s^2}{\mu^2}\right)$ for $\frac{1}{\epsilon}$. Integrating over l and carrying out the sum over n in (3.4) yields

$$A = -\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2\pi\theta m)}{m(4\pi m)^\epsilon}\;\Gamma(1+\epsilon)\,\zeta(1+\epsilon)\,. \tag{3.5}$$

For $\epsilon \ll 1$, we can expand

$$A = -\frac{1}{\epsilon}\left(\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2\pi\theta m)}{m}\right) + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\ln(4\pi m)}{m}\;\sin(2\pi\theta m) + O(\epsilon)\,, \tag{3.6}$$

from which the $\epsilon \to 0^+$ divergence is nicely read off. The term in parentheses is a standard example of a Fourier series. In the open interval $(0,1)$ it sums to $1/2-\theta$,

[3] This method of regularisation has already been put to work in [15], however with a slightly different result than ours. We will discuss the difference between this result and the result of [15] in the final section of the paper.

[4] Note the seemingly different formula for $\frac{\vartheta_1'}{\vartheta_1}$ as the one in [15]. The latter can be brought into the form displayed here by performing the sum over k there.

while for $\theta = 0$ it gives zero and elsewhere it sums to formulas given by 'periodic continuation' with period 1. Upon performing the sum over I in (3.1) and using (3.2) one finds that the $\frac{1}{\epsilon}$-term is multiplied by a constant, which, taking into account the prefactors in (3.1), is the contribution of brane stack b to the beta function. Thus, upon $\frac{1}{\epsilon} \to \ln\left(\frac{M_s^2}{\mu^2}\right)$, the correct one-loop running is reproduced. It remains to sum the left-over infinite series[5] in (3.6).

It turns out that, for $0 < \theta < 1$,

$$\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\ln(4\pi m)}{m}\sin(2\pi\theta m) = \frac{1}{2}\ln\left(\frac{\Gamma(\theta)}{\Gamma(1-\theta)}\right) - (\ln 2 - \gamma)\,(\theta - 1/2)\ , \tag{3.7}$$

where γ is the Euler–Mascheroni constant.

A way to derive the expression on the right hand side of this equation is presented in appendix A. Presently, let us verify that the relation (3.7) is true. The idea is to interpret the left hand side of (3.7) as the Fourier series of its right hand side (for the theory of Fourier series of functions with infinities like the one at hand see e.g. [22]).

The even terms in this Fourier series are all zero for reasons of symmetry[6], while for the odd terms we have to calculate the sine Fourier coefficients

$$b_m := 2\int_0^1 d\theta\, F(\theta)\, \sin(2\pi m\theta)\,, \tag{3.8}$$

for $m = 1, 2, 3, \dots$, with $F(\theta) := \frac{1}{2}\ln\left(\frac{\Gamma(\theta)}{\Gamma(1-\theta)}\right) - (\ln 2 - \gamma)\,(\theta - 1/2)$.

The only non-trivial integrals arising in this computation are those from the first term in $F(\theta)$

$$J_m := \int d\theta\, \ln\left(\frac{\Gamma(\theta)}{\Gamma(1-\theta)}\right)\, \sin(2\pi m\theta)\,, \tag{3.9}$$

while the second term contributes $(\ln 2 - \gamma)/(m\pi)$.

In order to proceed, we employ the expansion

$$\ln(\Gamma(\theta)) = -\gamma\,\theta - \ln(\theta) + \sum_{k=1}^{\infty}\left[\frac{\theta}{k} - \ln\left(1 + \frac{\theta}{k}\right)\right]\ . \tag{3.10}$$

[5] By Dirichlet's test, it converges for all θ in, say, the open interval $(0, 1)$. Moreover, it converges trivially to zero for $\theta = 0$, and therefore it converges for all θ by periodicity.

[6] The rhs of (3.7) is odd under reflection of θ in $1/2$, i.e. under $\theta \to 1 - \theta$.

With this, the integrals J_m are easily calculated:

$$
\begin{aligned}
J_m &= -2\gamma \int_0^1 d\theta\, \theta \sin(2\pi m\theta) + \int_0^1 d\theta\, \sin(2\pi m\theta)\, \ln\left(\frac{1-\theta}{\theta}\right) + \\
&\quad + \sum_{k=1}^{\infty} \int_0^1 d\theta\, \theta\, \sin(2\pi m\theta)\, \left(\frac{2\,\theta+1}{k} + \ln\left(\frac{k+1-\theta}{k+\theta}\right)\right) \\
&= \frac{\gamma}{m\pi} + \frac{\gamma - \mathrm{Ci}[2\pi m] + \ln(2\pi m)}{m\pi} + \\
&\quad + \frac{1}{m\pi}\left[\sum_{k=1}^{\infty}\Big(\mathrm{Ci}[2km\pi] - \mathrm{Ci}[2(k+1)m\pi]\Big) - \sum_{k=1}^{\infty}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right)\right],
\end{aligned} \tag{3.11}
$$

where Ci is the cosine integral. Almost all terms in the sums cancel so that one eventually obtains the simple expression

$$
\begin{aligned}
J_m &= \frac{1}{m\pi}\Big[2\gamma + \ln(2\pi m) - \lim_{N\to\infty} \mathrm{Ci}[2(N+1)m\pi] - (\ln\Gamma(1) + \gamma + \ln 1)\Big] \\
&= \frac{\gamma + \ln(2\pi m)}{m\pi}\,.
\end{aligned} \tag{3.12}
$$

Therefore, collecting terms, we find

$$
b_m = \frac{\gamma + \ln(2\pi m)}{m\pi} + \frac{\ln 2 - \gamma}{m\pi} = \frac{\ln(4\pi m)}{\pi m}\,, \tag{3.13}
$$

as was to be shown.

The upshot of this discussion is that we have regularised, for $0 < \theta < 1$, (suppressing the tadpole and $\ln(M_s^2/\mu^2)$ terms)

$$
-\frac{1}{\pi}\int_0^\infty dl\, \frac{\vartheta_1'}{\vartheta_1}(\theta, 2il) \rightarrow \frac{1}{2}\ln\left(\frac{\Gamma(\theta)}{\Gamma(1-\theta)}\right) - (\ln 2 - \gamma)\,(\theta - 1/2)\ . \tag{3.14}
$$

Now, in view of the angle condition (3.2), it is inevitable that some θs are going to be negative, so that we also have to consider the case $-1 < \theta < 0$. But this is easily reduced to the already derived formulas, with the result (put $-\theta =: \tilde{\theta} > 0$ and apply (3.7)):

$$
-\frac{1}{\pi}\int_0^\infty dl\, \frac{\vartheta_1'}{\vartheta_1}(\theta, 2il) \rightarrow \frac{1}{2}\ln\left(\frac{\Gamma(1+\theta)}{\Gamma(-\theta)}\right) - (\ln 2 - \gamma)\,(\theta + 1/2)\ , \tag{3.15}
$$

for $-1 < \theta < 0$.

Now we are finally in a position to write down the complete regularised annulus amplitude (3.1). The result is (still suppressing the tadpole):

$$
\begin{aligned}
T^A(\mathrm{D6}_a, \mathrm{D6}_b) = \frac{I_{ab}N_b}{2}\Bigg[&\ln\left(\frac{M_s^2}{\mu^2}\right)\sum_{I=1}^{3}\mathrm{sign}(\theta_{ab}^I) - \\
&- \ln\prod_{I=1}^{3}\left(\frac{\Gamma(|\theta_{ab}^I|)}{\Gamma(1-|\theta_{ab}^I|)}\right)^{\mathrm{sign}(\theta_{ab}^I)} - \sum_{I=1}^{3}\mathrm{sign}(\theta_{ab}^I)\,(\ln 2 - \gamma)\Bigg]\,,
\end{aligned} \tag{3.16}
$$

where we have taken into account the angle condition (3.2).

The calculation of the Möbius diagrams,

$$
\begin{aligned}
T^M(\mathrm{D6}_a, \mathrm{O6_k}) &= \pm \frac{i4I_{a;O6_k}}{\pi} \int_0^\infty \frac{dt}{t} \sum_{I=1}^{3} \frac{\vartheta_1'}{\vartheta_1}\left(i\theta^I_{a;O6_k} t, \tfrac{it}{2} + \tfrac{1}{2}\right) \\
&= \pm \frac{4I_{a;O6_k}}{\pi} \int_0^\infty dl \sum_{I=1}^{3} \frac{\vartheta_1'}{\vartheta_1}\left(\theta^I_{a;O6_k}, 2il - \tfrac{1}{2}\right), \qquad (3.17)
\end{aligned}
$$

proceeds in a rather similar fashion. Here, $I_{a;O6_k}$ is the intersection number of the D-brane and the orientifold plane k, $\theta^I_{a;O6_k}$ is the intersection angle of brane a and the orientifold plane k on the I'th torus and $l = 1/(4t)$. The additional summand $-\frac{1}{2}$ in the second argument of the theta functions in (3.17) leads to an additional $(-1)^{mn} = \frac{1}{2}(1 + (-1)^n + (-1)^m - (-1)^{m+n})$ in the expression corresponding to (3.4). Eventually, one finds

$$
\begin{aligned}
T^M(\mathrm{D6}_a, \mathrm{O6_k}) &= \pm I_{a;O6_k} \sum_{I=1}^{3} \Bigg[4\,\theta^I_{a;O6_k} \left(-\ln\left(\frac{M_s^2}{\mu^2}\right) + 2\ln 2 - \gamma \right) \qquad (3.18) \\
&\qquad + \ln\left(\frac{M_s^2}{\mu^2}\right) f(\theta^I_{a;O6_k}) + g(\theta^I_{a;O6_k}) \Bigg],
\end{aligned}
$$

where the first term vanishes after imposing the supersymmetry condition,

$$
f(\theta) = \begin{cases} \mathrm{sign}(\theta) & \text{for } -\frac{1}{2} < \theta < \frac{1}{2} \\ -3 & \text{for } -1 < \theta < -\frac{1}{2} \\ 3 & \text{for } \quad \frac{1}{2} < \theta < 1, \end{cases} \qquad (3.19)
$$

and

$$
g(\theta) = \begin{cases} (\gamma - 3\ln 2)\,\mathrm{sign}(\theta) - \mathrm{sign}(\theta) \ln\left(\frac{\Gamma(2|\theta|)}{\Gamma(1-2|\theta|)}\right) & \text{for } -\frac{1}{2} < \theta < \frac{1}{2} \\ -3\gamma + 5\ln 2 + \ln\left(\frac{\Gamma(-2\theta-1)}{\Gamma(2+2\theta)}\right) & \text{for } -1 < \theta < -\frac{1}{2} \\ 3\gamma - 5\ln 2 - \ln\left(\frac{\Gamma(2\theta-1)}{\Gamma(2-2\theta)}\right) & \text{for } \quad \frac{1}{2} < \theta < 1. \end{cases} \qquad (3.20)
$$

The entire one loop corrections to the gauge coupling on brane stack a is then given by the sum over all annulus and Möbius diagrams with one boundary on brane a.

Cases where the intersection angles sum to $\pm 2n$, $n \in \mathbb{N}^*$, can be treated by periodic continuation of our formulas (cf. (3.6) and (3.7)).

4 Discussion and relation to previous work

In order to compare the derived results to [15], it is useful to specialise to $\theta_{ab}^{1,2} > 0$, $\theta_{ab}^3 < 0$. Equation (3.16) then contains the following threshold corrections

$$\Delta_a = -\frac{b_a}{16\pi^2} \ln\left[\frac{\Gamma(\theta_{ab}^1)\Gamma(\theta_{ab}^2)\Gamma(1+\theta_{ab}^3)}{\Gamma(1-\theta_{ab}^1)\Gamma(1-\theta_{ab}^2)\Gamma(-\theta_{ab}^3)}\right], \tag{4.1}$$

($b_a = \frac{I_{ab}N_b}{2}$), which are to be compared with [15]

$$\tilde{\Delta}_a = -\frac{b_a}{16\pi^2} \ln\left[\frac{\Gamma(1+\theta_{ab}^1)\Gamma(1+\theta_{ab}^2)\Gamma(1+\theta_{ab}^3)}{\Gamma(1-\theta_{ab}^1)\Gamma(1-\theta_{ab}^2)\Gamma(1-\theta_{ab}^3)}\right]. \tag{4.2}$$

Clearly, Δ_a and $\tilde{\Delta}_a$ are not identical, the difference being

$$\Delta_a - \tilde{\Delta}_a = -\frac{b_a}{16\pi^2} \ln\left[-\frac{\theta_{ab}^3}{\theta_{ab}^1\theta_{ab}^2}\right]. \tag{4.3}$$

This difference appears to stem from the different treatment of open string states in the threshold corrections, which are located at the intersection of two D6-branes and whose masses are proportional to an integer multiple of the intersection angle θ_{ab}^I. This interpretation will be motivated in appendix B. These states are in fact included in the threshold corrections Δ_a. For small intersection angles some of these states become lighter than the string scale M_s, and hence Δ_a logarithmically diverges for $\theta_{ab}^I \to 0$. On the other hand, $\tilde{\Delta}_a$ is completely regular for $\theta_{ab}^I \to 0$, because it does not contain the contribution of these states that become light when $\theta_{ab}^I \to 0$.[7] In more technical terms, this different behavior can be traced back to how the infrared divergences were treated during the computation of the threshold corrections. In the present work, the contribution of the massless modes appears in the logarithmic running of the gauge coupling constant, whereas the contribution of the modes that become light for $\theta_{ab}^I \to 0$ is kept in Δ_a. This is in contrast to the infrared regularisation method employed in [15] for the computation of $\tilde{\Delta}_a$, where also the contribution of the modes with masses proportional to $m\theta_{ab}^I$, $m \in \mathbb{N}$ is subtracted from the threshold corrections.

Finally, let us remark that the one-loop correction Δ_a to the gauge coupling constant is not the real part of a holomorphic function when expressed in terms of the complex structure moduli fields U^I of the underlying torus $\mathbb{T}^6$, since the intersection angles θ_{ab}^I are non-holomorphic functions of the U^I. The reason for this non-holomorphy due to σ-model anomalies and other issues of holomorphy in the context of instanton corrections to the effective action of intersecting D-brane models are discussed in [23].

[7] Note however that both Δ_a and $\tilde{\Delta}_a$ contain the contribution of states that become light for $\theta_{ab}^I \to 1$.

Acknowledgements

We would like to thank Emilian Dudas, Michael Haack, Sebastian Moster, Erik Plauschinn, Timo Weigand and especially Stephan Stieberger for valuable discussions. This work is supported in part by the European Community's Human Potential Programme under contract MRTN-CT-2004-005104 'Constituents, fundamental forces and symmetries of the universe'.

A Loop channel calculation and zeta-function regularisation

As promised in the main text, in this appendix we determine the analytic form of the threshold corrections using zeta-function regularisation in the loop channel. The threshold corrections up to an additive constant, i.e. in particular the full moduli dependence, will be computed. For the annulus diagram, this means evaluating

$$T^A(\mathrm{D6}_a, \mathrm{D6}_b) \;=\; \frac{iI_{ab}N_b}{2\pi} \int_0^\infty \frac{dt}{t} \sum_{I=1}^{3} \frac{\vartheta_1'}{\vartheta_1}\left(\tfrac{i\theta^I_{ab}t}{2}, \tfrac{it}{2}\right). \tag{A.1}$$

Using the product representation of the theta function and the Taylor expansion $\ln(1-z) = -\sum_{n=1}^\infty \frac{z^n}{n}$ one can derive

$$\frac{\vartheta_1'}{\vartheta_1}(\nu,\tau) = \frac{\partial}{\partial\nu}\ln\vartheta_1(\nu,\tau) = \pi\cot(\pi\nu) - 4\pi i \sum_{m,n=1}^{\infty} e^{2\pi i\tau mn}\sinh(2\pi i\nu n), \tag{A.2}$$

which is valid if $|\exp(2\pi i(\tau n \pm \nu))| < 1$ for all $n \in \mathbb{N}$. Using

$$\coth(x) = \mathrm{sign}(x)[1 + 2\sum_{n=1}^{\infty}\exp(-2|x|m)] \tag{A.3}$$

and extracting the divergence for $t \to \infty$ stemming from the massless open string modes, one finds

$$\begin{aligned}
\tilde{\Delta} \;&=\; \int_0^\infty \frac{dt}{t}\frac{\vartheta_1'}{\vartheta_1}\left(\frac{i\theta t}{2}, \frac{it}{2}\right) + \int_1^\infty \frac{dt}{t}\,\pi i\;\mathrm{sign}(\theta) \qquad\qquad (A.4)\\
&=\; -2\pi i\,\mathrm{sign}(\theta)\int_0^\infty \frac{dt}{t}\sum_{n,m=1}^{\infty}\Big[\exp(-\pi t n(m-1+|\theta|)) - \exp(-\pi t n(m-|\theta|))\Big]\\
&\quad\; -\int_0^1 \frac{dt}{t}\pi i\,\mathrm{sign}(\theta)\\
&=\; -2\pi i\,\mathrm{sign}(\theta)\sum_{n,m=1}^{\infty}\ln\left(\frac{\pi n(m-|\theta|)}{\pi n(m-1+|\theta|)}\right) - \pi i\,\mathrm{sign}(\theta)\lim_{N\to\infty}\ln N\,.
\end{aligned}$$

Clearly, the sum over the positive integers n is divergent, which was expected, as we have not yet deducted the ultraviolet divergence due to the tadpole. The main observation is that performing a simple zeta-function regularisation $\sum_{n=1}^{\infty} 1 = \zeta(0) = -\frac{1}{2}$ seems to take precisely care of the tadpole. Indeed after zeta-function regularisation we get

$$\tilde{\Delta} \;=\; \pi i\,\mathrm{sign}(\theta) \sum_{m=1}^{\infty} \left[\ln\left(1 - \frac{|\theta|}{m}\right) - \ln\left(1 - \frac{1-|\theta|}{m}\right) \right] - \pi i\,\mathrm{sign}(\theta) \lim_{N\to\infty} \ln N. \tag{A.5}$$

Using the relations

$$\ln\Gamma(1+x) = -\gamma x + \sum_{k=1}^{\infty} \left[\frac{x}{k} - \ln(1+\frac{x}{k})\right] \quad \text{and } \gamma = \lim_{N\to\infty} \left(\sum_{k=1}^{N} \frac{1}{k} - \ln N \right) \tag{A.6}$$

the last expression becomes

$$\tilde{\Delta} = \pi i\,\mathrm{sign}(\theta) \ln\left(\frac{\Gamma(|\theta|)}{\Gamma(1-|\theta|)} \right) - 2\pi i\,\theta \lim_{N\to\infty} \ln N. \tag{A.7}$$

Finally, performing the sum over I yields:

$$\begin{aligned} T^A(\mathrm{D6}_a, \mathrm{D6}_b) = &\frac{I_{ab} N_b}{2} \sum_{I=1}^{3} \mathrm{sign}(\theta^I_{ab}) \int_1^{\infty} \frac{dt}{t} \\ &- \frac{I_{ab} N_b}{2} \ln \prod_{I=1}^{3} \left(\frac{\Gamma(|\theta^I_{ab}|)}{\Gamma(1-|\theta^I_{ab}|)} \right)^{\mathrm{sign}(\theta^I_{ab})} + I_{ab} N_b \left(\sum_{I=1}^{3} \theta^I_{ab} \right) \lim_{N\to\infty} \ln(N) \end{aligned} \tag{A.8}$$

The last term vanishes due to the supersymmetry condition (3.2). Thus, indeed, the calculation in the loop channel using zeta-function regularisation gives the same result, up to a constant, as the one in the tree channel after extracting the divergence that cancels due to the tadpole condition, as has been done in section 3. Zeta-function regularisation seems to correctly subtract the divergence due to the closed string tadpole, an observation which we believe to be valuable as a heuristic device. Furthermore, the Möbius diagram can be dealt with analogously if $|\theta| < \frac{1}{2}$.

B On light modes in the threshold corrections

The purpose of this appendix is to provide evidence for the statement that the difference between the result derived here and in [15] is due to the treatment of open string modes whose masses are given by an integer multiple of an intersection angle.

Note first that if one (when working in loop channel as in Appendix A) extracts not only the constant term from the (hyperbolic) cotangent, as is done in (A.4), but the entire cotangent, one arrives at the result for the threshold corrections derived in [15]. One is thus led to the following hypothesis: Zeta-function regularisation in the loop channel is equivalent to extracting the cotangent from the expansion of $\frac{\vartheta_1'}{\vartheta_1}$ in (A.2) in the tree channel (this was essentially proven in this work), whereas zeta-function regularisation in the tree channel (which was done in [15]) is equivalent to extracting the cotangent term in the loop channel[8].

Thus, the difference between the present results and the ones in [15] appears to stem from the term

$$\int \frac{dt}{t}\pi \left[\coth\left(\frac{\pi\theta t}{2}\right) - \mathrm{sign}(\theta)\right] = 2\pi\,\mathrm{sign}(\theta)\int \frac{dt}{t}\sum_{n=1}^{\infty}\exp(-\pi|\theta|tn)\,, \qquad \text{(B.1)}$$

which can be interpreted as the contribution of modes with masses given by $n\theta$, $n \in \mathbb{N}$. Regularising the ultraviolet divergence in (B.1) one finds

$$\int \frac{dt}{t}\sum_{n=1}^{\infty}\Big(e^{-\pi|\theta|tn} - e^{-\pi N tn}\Big) = \zeta(0)\Big(-\ln|\theta| + \ln N\Big)\,. \qquad \text{(B.2)}$$

The finite, moduli-dependent term thus precisely accounts for the difference in (4.3).

[8]Note, when performing the entire computation in loop-channel, one still has to employ zeta-function regularisation for the substraction of the tree-channel tadpoles.

References

[1] A. M. Uranga, “Chiral four-dimensional string compactifications with intersecting D-branes,” *Class. Quant. Grav.* **20** (2003) S373–S394, `hep-th/0301032`.

[2] D. Lüst, “Intersecting brane worlds: A path to the standard model?,” *Class. Quant. Grav.* **21** (2004) S1399–1424, `hep-th/0401156`.

[3] R. Blumenhagen, M. Cvetic, P. Langacker, and G. Shiu, “Toward realistic intersecting D-brane models,” *Ann. Rev. Nucl. Part. Sci.* **55** (2005) 71–139, `hep-th/0502005`.

[4] R. Blumenhagen, B. Körs, D. Lüst, and S. Stieberger, “Four-dimensional string compactifications with D-branes, orientifolds and fluxes,” `hep-th/0610327`.

[5] R. Blumenhagen, M. Cvetic, and T. Weigand, “Spacetime instanton corrections in 4D string vacua - the seesaw mechanism for D-brane models,” *Nucl. Phys.* **B771** (2007) 113–142, `hep-th/0609191`.

[6] S. A. Abel and M. D. Goodsell, “Realistic Yukawa couplings through instantons in intersecting brane worlds,” `hep-th/0612110`.

[7] N. Akerblom, R. Blumenhagen, D. Lüst, E. Plauschinn, and M. Schmidt-Sommerfeld, “Non-perturbative SQCD Superpotentials from String Instantons,” *JHEP* **04** (2007) 076, `hep-th/0612132`.

[8] M. Billo *et al.*, “Classical gauge instantons from open strings,” *JHEP* **02** (2003) 045, `hep-th/0211250`.

[9] L. E. Ibanez and A. M. Uranga, “Neutrino Majorana masses from string theory instanton effects,” *JHEP* **03** (2007) 052, `hep-th/0609213`.

[10] M. Bianchi and E. Kiritsis, “Non-perturbative and Flux superpotentials for Type I strings on the Z3 orbifold,” `hep-th/0702015`.

[11] M. Cvetic, R. Richter, and T. Weigand, “Computation of D-brane instanton induced superpotential couplings: Majorana masses from string theory,” `hep-th/0703028`.

[12] R. Argurio, M. Bertolini, G. Ferretti, A. Lerda, and C. Petersson, “Stringy Instantons at Orbifold Singularities,” `arXiv:0704.0262 [hep-th]`.

[13] M. Bianchi, F. Fucito, and J. F. Morales, “D-brane Instantons on the T6/Z3 orientifold,” `arXiv:0704.0784 [hep-th]`.

[14] L. E. Ibanez, A. N. Schellekens, and A. M. Uranga, "Instanton Induced Neutrino Majorana Masses in CFT Orientifolds with MSSM-like spectra," arXiv:0704.1079 [hep-th].

[15] D. Lüst and S. Stieberger, "Gauge threshold corrections in intersecting brane world models," *Fortsch. Phys.* **55** (2007) 427–465, hep-th/0302221.

[16] C. Bachas and C. Fabre, "Threshold Effects in Open-String Theory," *Nucl. Phys.* **B476** (1996) 418–436, hep-th/9605028.

[17] I. Antoniadis, C. Bachas, and E. Dudas, "Gauge couplings in four-dimensional type I string orbifolds," *Nucl. Phys.* **B560** (1999) 93–134, hep-th/9906039.

[18] M. Berg, M. Haack, and B. Körs, "Loop corrections to volume moduli and inflation in string theory," *Phys. Rev.* **D71** (2005) 026005, hep-th/0404087.

[19] M. Bianchi and E. Trevigne, "Gauge thresholds in the presence of oblique magnetic fluxes," *JHEP* **01** (2006) 092, hep-th/0506080.

[20] P. Anastasopoulos, M. Bianchi, G. Sarkissian, and Y. S. Stanev, "On gauge couplings and thresholds in type I gepner models and otherwise," *JHEP* **03** (2007) 059, hep-th/0612234.

[21] V. S. Kaplunovsky, "One loop threshold effects in string unification," hep-th/9205070.

[22] E. W. Hobson, "The Theory of Functions of a Real Variable and the Theory of Fourier's Series, Vol. 2,". New York: Dover Publications (1957) 2nd edition.

[23] N. Akerblom, R. Blumenhagen, D. Lüst, and M. Schmidt-Sommerfeld, "Instantons and Holomorphic Couplings in Intersecting D-brane Models," to appear.